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Centralizers and Entropy

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Abstract. We prove that for a large class of bidimensional real analytic diffeomorphisms the centralizer is trivial: they only commute with their own integer powers. In particular this property holds for an open and dense subset of those having positive topological entropy.

1. Introduction

Given a compact, connected and boundaryless two-dimensional Riemannian manifold, M, we consider the space of real analytic diffeomorphisms on M, $\text{Diff}^w(M)$, endowed with the usual c^k*-topology*, $k \in \mathbb{N} \cup \{\infty\}$ or $k = w$, (for the definition of the c^w*-topology* and related properties we refer the reader to [**BT**] or [**R**]).

The *centralizer* of $f \in \text{Diff}^w(M)$, denoted by $Z^w(f)$, is the subset of real analytic diffeomorphisms that commute with f, that is

$$Z^w(f) = \{g \in \text{Diff}^w(M); f \circ g = g \circ f\}.$$

We say that the centralizer is *trivial* if it reduces to the own integer powers of f.

Palis and Yoccoz ([**PY**]) proved that, in the space of diffeomorphisms that satisfy Axiom A and the Transversality Condition, to have trivial centralizer is a *generic* property, that is for a c^∞-open and dense subset the centralizer is trivial. Later the author ([**R**]), conjugating Palis and Yoccoz work with Broer and Tangerman technics ([**BT**]) that allow to get real analytic perturbations (in the c^w-topology) from the usual c^∞ ones, transpose their result to the real analytic context maintaining the

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dynamical conditions.

In this paper we show that the same result is true (without dynamical assumptions) for a large class of bidimensional real analytic diffeomorphisms.

In order to state our results let us define

$$U_0 = \{f \in \mathrm{Diff}^w(M)\colon h_{top}(f) > 0\}$$
$$U_1 = \mathrm{Int}_{c^1}(\{f \in \mathrm{Diff}^w(M)\colon h_{top}(f) = 0\})$$

where $\mathrm{Int}_{c^1}(A)$ denotes the c^1-interior of A and $h_{top}(f)$ is the topological entropy of f. Note that U_0 is c^1-open (Katok, [**K**]) and $U = U_0 \cup U_1$ is c^k-open (for all k) and c^1-dense in $\mathrm{Diff}^w(M)$.

Here we prove:

Theorem 1. *There exists U', a c^∞-open and c^1-dense subset of U, whose elements have trivial centralizer.*

To obtain this result we first observe that, by Katok's caracterization of positive topological entropy of $c^{1+\alpha}$ two-dimensional diffeomorphisms ([**K**]), if $f \in U_0$ then there exist transversal homoclinic points associated to a saddle, and that if $f \in U_1$, as a consequence of a recent result of Araújo and Mañé ([**AM**]), then it can be c^1 approximated by a diffeomorphism exhibiting a sink and a source whose basins have non-empty intersection. Then we prove that

(i) if $f \in U_0 \cap U'$ and $h \in Z^w(f)$ then $h \circ f^i|_{W^s(P_f)\cup W^u(P_f)} \equiv \mathrm{Id}$, for some $i \in Z$, where P_f is a saddle point belonging to a hyperbolic horseshoe,

(ii) if $f \in U_1 \cap U'$ and $h \in Z^w(f)$ then $h \circ f^i|_{W^s(P_f)\cup W^u(Q_f)} \equiv \mathrm{Id}$, for some $i \in Z$, where $P_f(Q_f)$ is a sink (source) and $W^s(P_f) \cap W^u(Q_f) \neq \varnothing$,

(iii) if f and h satisfy (i) or (ii) then $h \circ f^i \equiv \mathrm{Id}$.

We point out that analiticity is only required in order to obtain (iii).

Actually the set U' we get in Theorem 1 is c^k-open and dense in U_0 ($k \in \mathbb{N} \cup \{\infty\}$ or $k = w$). Moreover it remains open to prove (or disprove) that U is c^k dense in $\mathrm{Diff}^k(M)$, for k different from one.

Finally we consider the *Conservative Hénon family*,

$$f_\rho : \mathbb{R}^2 \to \mathbb{R}^2, f_\rho(x,y) = (\rho - x^2 - y, x), \rho \in \mathbb{R},$$

and, as a consequence of (i) and of the geometry of this one-parameter family, prove

Theorem 2. *$Z^w(f_\rho)$ is trivial for all $\rho > -1$.*

2. Proof of Theorem 1 in U_0

Let us fix f in U_0 and V a c^k neighbourhood of f, $k \in \mathbb{N} \cup \{\infty\}$ or $k = w$. First we observe that, as f has positive topological entropy, there is a hyperbolic periodic point, say P_f, such that $W^s(P_f)$ and $W^u(P_f)$ have a point of transversal intersection, Q_f.

Let $V_1 \subseteq V$ be a c^k neighbourhood of f such that for all $g \in V_1$ the analytic continuations of P_f and Q_f are well defined, P_g and Q_g respectively; let V_2 be a c^k open and dense subset of V_1 such that all the periodic points of $g \in V_2$ with period less or equal to r (the period of P_g) are hyperbolic and if P_1 and P_2 are periodic with the same period $s \leq r$ then P_2 belongs to the g-orbit of P_1 or $Dg^s(P_1)$ and $Dg^s(P_2)$ are not conjugated in the space of linear isomorphisms.

The proof of the Theorem in U_0 follows from forthcomig Proposition.

Proposition 1. *There exists a c^k open and dense subset of V_2, W, such that if $g \in W$ and $h \in Z^w(g)$ then, for some $i \in Z$, $g^i \circ h(x) = x$ for all $x \in W^s(P_g) \cup W^u(P_g)$.*

In fact if $g \in W$ and $h \in Z^w(g)$ then $h \circ g^i|_{W^s(P_g) \cup W^u(P_g)} \equiv \mathrm{Id}$ for some $i \in Z$, and, using, λ-*lemma* and the analicity of h, we conclude that for any transversal section L of $W^s(P_g)$, $h \circ g^i|_L \equiv \mathrm{Id}$, which clearly implies that the centralizer of g is trivial.

Proof of Proposition 1. Since the arguments presented here are similar to those introduced in [**PY**] and used in [**R**] and [**R1**], the details are omited.

Given $g \in V_2$ there is a c^k neighbourhood of g, $V_g \subseteq V_2$, such that

(i) for all $g_1 \in V_g$ there are c^∞ linearizations of

$$g_1^r|_{W^s(P_{g_1^r})} \quad \text{and} \quad g_1^r|_{W^u(P_{g_1^r})},$$

say $\varphi^s_{g_1}$ and $\varphi^u_{g_1}$ respectively,

(ii) the maps $\psi^\sigma\colon (V_g, c^k) \to (C^\infty, c^{k-1})$, $\psi^\sigma(g_1) = \varphi^\sigma_{g_1}$, $\sigma = s$ or u, are continuous.

Now, given $g_1 \in V_g$ and $h \in Z^w(g_1)$ we have that $h(P_{g_1})$ belongs to the g_1-orbit of P_{g_1}; thus there exists $j \in \mathbb{Z}$ such that $h' = h \circ g_1^j$ fixes P_{g_1}. Define $h^\sigma_{g_1} = (\varphi^\sigma_{g_1})^{-1} \circ h' \circ \varphi^\sigma_{g_1}$, $\sigma = s$ or u. As $h^\sigma_{g_1}$ are c^∞ and commute with $A^\sigma_{g_1}$ ($A^\sigma_{g_1} = (\varphi^\sigma_{g_1})^{-1} \circ g_1^r \circ \varphi^\sigma_{g_1}$) they are linear maps. Writing $A^\sigma_{g_1}(x) = \lambda^\sigma_{g_1} \cdot x$ and $h^\sigma_{g_1}(x) = \mu^\sigma_h \cdot x$ we define

$$\alpha^\sigma_{g_1,h} = \frac{\log|\mu^\sigma_h|}{\log\left|\lambda^\sigma_{g_1}\right|}, \quad \sigma = s \text{ or } u.$$

It is not difficult to prove that $\alpha^s_{g_1,h} = \alpha^u_{g_1,h} \in \mathbb{Q}$, see for instance [**R1**].

Therefore if $g_1 \in V_g$ then its centralizer can be identified with a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Q}$,

$$h \in Z^w(g_1) \longmapsto R(h) = (\theta_{s,h}, \theta_{u,h}, \alpha_{g_1,h}),$$

where $\theta_{s,h} = \dfrac{|\mu^\sigma_h|}{\mu^\sigma_h}$, $\sigma = s$ or u. Observe that

$$R(g_1) = \left(\frac{\left|\lambda^s_{g_1}\right|}{\lambda^s_{g_1}}, \frac{\left|\lambda^u_{g_1}\right|}{\lambda^u_{g_1}}, 1 \right)$$

and that $R(h) = R(g_1)$ implies that $h' = h \circ g_1^{j-r}$ is the identity on $W^s(P_{g_1}) \cup W^u(P_{g_1})$ (the converse is trivial). An element $(\theta_s, \theta_u, \frac{p}{q})$ is called a *root of order* q of g_1 if $(p,q) = 1$, $(\theta_s, \theta_u, \frac{p}{q}) \neq R(g_1)$ and there is some $h \in Z^w(g_1)$ such that $R(h) = (\theta_s, \theta_u, \frac{p}{q})$. It is easy to see that if $(\theta_s, \theta_u, \frac{p}{q})$ is a root of g_1 then there are θ'_u, $\theta'_s \in \mathbb{Z}_2$ such that $(\theta'_s, \theta'_u, \frac{1}{q})$ is a root of g_1.

Now we fix $(\theta_s, \theta_u, \frac{1}{q})$ and assume that it is a root of some $g_0 \in V_g$. There exists a transversal homoclinic point associated to P_{g_0}, say

$Y_{g_0}(Y_{g_0} = h(Q_{g_0}))$, where $h \in \mathbb{Z}^w(g_0)$ and $R(h) = (\theta_s, \theta_u, \frac{1}{q})$ such that

$$(\varphi^\sigma_{g_0})^{-1}(Y_{g_0}) = \theta_\sigma \left|\lambda^\sigma_{g_0}\right|^{\frac{1}{q}} (\varphi^\sigma_{g_0})^{-1}(Q_{g_0}), \ \sigma = s \text{ or } u. \qquad (*)$$

Also, as $(\theta_s, \theta_u, \frac{1}{q})$ is different from $R(g_0)$, it follows that Y_{g_0} does not belong to the g_0-orbit of Q_{g_0}. Now g_0 can approximated (in the c^k topology) by $g_1 \in \mathrm{Diff}^w(M)$ such that condition (*) is not satisfied. Finally continuity of ψ^σ, Q_{g_0}, Y_{g_0} imply that there is a neighbourhood of g_1, say W_q, where (*) is not satisfied, that is $(\theta_s, \theta_u, \frac{1}{q})$ is not a root for all g_1 in W_q.

Remark that there exists $\delta > 0$ such that there are no roots of order q_1 in W_q if

$$\left|\frac{p_1}{q_1} - \frac{p}{q}\right| < \delta.$$

Thus there exists $q_0 \in \mathbb{N}$ such that if $g \in W_q$ and if $h \in Z^w(g)$ then h is a root of order less or equal to q_0.

Therefore we just have to repeat the above argument a finite number of times in order to obtain a c^k-open and dense subset of W_q whose elements have trivial centralizer, thus ending the proof. □

3. Proof of Theorem 1 in U_1

In order to get some dynamical properties for $f \in U_1$ let us refer the following result of Araújo and Mañé.

Theorem ([AM]). *If f is a c^2 bidimensional diffeomorphism such that all periodic points are hyperbolic then one of the following situations occurs:*

(i) there are a finite number of hyperbolic attractors, say $\Lambda_1, \dots, \Lambda_k$ and a finite number of contracting irrational rotations, say $\Lambda_{k+1}, \dots, \Lambda_n$, such that $\cup_{i=1}^n W^s(\Lambda_i)$ has full Lebesgue measure;

(ii) f can be c^1 approximated by a diffeomorphism exhibiting a (dissipative) homoclinic tangency.

From this Theorem it is not difficult to prove that the set U_2, consisting of those diffeomorphisms that have a sink P and a source Q with $W^s(P) \cap W^u(Q) \neq \varnothing$, is c^1-open and dense in U_1. Therefore we just have to prove that given any c^1-neighbourhood V_0 of $f_0 \in U_2$ there exists a c^∞-open set $U \subseteq V_0$ such that if $f \in U$ and $h \in Z^w(f)$ then, for some $s \in \mathbb{Z}$, $h \circ f^s|_{W^s(P_f)} \equiv \mathrm{Id}$, where P_f denotes the analytic continuation of P_{f_0}.

Let us fix $f_0 \in U_2$ and V_0 an arbitrary neighbourhood of f_0; we denote by k_p, respectively k_q, the period of P_{f_0}, respectively Q_{f_0}, and define $k_0 = m.m.c.\{k_p, k_q\}$. If V_0 is small then we can choose V_1, c^1-open and dense in V_0, such that all $g \in V_1$ satisfy

(i) all the periodic points with period less or equal to k_0 are hyperbolic;

(ii) the eigenvalues of $Dg_P^{k_p}$ and $Dg_Q^{k_q}$ have multiplicity one and are non-resonant, where P and Q are the analitic continuations of P_{f_0} and Q_{f_0} respectively;

(iii) if x, $y \in Per(g)$ have period t, less or equal to k_0, then $y \in O_g(x)$ or Dg_x^t and Dg_y^t are not conjugated in the space of linear isomorphisms;

(iv) $W^s(P) \cap W^u(Q) \neq \varnothing$.

Now we fix any g in V_1; there is a c^∞-open neighbourhood V_2 of g such that

(v) for all $f \in V_2$ there are c^∞ linearizations of $f^{k_p}|_{W^s(P)}$ and $f^{k_q}|_{W^u(Q)}$, say φ_f^s and φ_f^u respectively; moreover the maps

$$\psi^\sigma \colon (V_2, c^\infty) \to (C^\infty, c^\infty), \psi^\sigma(f) = \varphi_f^\sigma, \sigma = s \text{ or } u,$$

are continuous.

From now on we assume that $f \in V_2$. We also suppose that the eigenvalues of $Df_P^{k_p}$ and of $Df_Q^{k_q}$ are real; the proof we sketch in this situation also works, with slight modifications, in the other cases (see, for instance, [**R1**]).

If $h \in Z^w(f)$ the condition (iii) implies that there are i, $j \in \mathbb{Z}$ such that $h \circ f^i(P) = P$ and $h \circ f^j(Q) = Q$. Let us define $h' = h \circ f^i$, $h'' = h \circ f^j$, and

$$h_1 = (\varphi_f^s)^{-1} \circ h' \circ \varphi_f^s,\ A_1 = (\varphi_f^s)^{-1} \circ f^{k_0} \circ \varphi_f^s,$$
$$h_2 = (\varphi_f^u)^{-1} \circ h'' \circ \varphi_f^u,\ A_2 = (\varphi_f^u)^{-1} \circ f^{k_0} \circ \varphi_f^u.$$

As $h_i \circ A_i = A_i \circ h_i$ and h_i is smooth it is easy to conclude that h_i is linear, $i \in \{1, 2\}$ (see, for instance, [**Ko**]).

Let us consider the case $i = j$, that is $h' \equiv h''$ or, more generally, consider

$$Z_F^w(f) = \{h \in Z^w(f);\ \text{exists } i \in \mathbb{Z} \text{ s.t. } h \circ f^i(P) = P \text{ and } h \circ f^i(Q) = Q\}.$$

It is not difficult to reduce the general situation to this case.

As before we define

$$\alpha_i = \frac{\log|\mu_i|}{\log|\lambda_i|},\ \alpha_i' = \frac{\log|\mu_i'|}{\log|\lambda_i'|},\ \theta_i = \frac{\lambda_i}{|\lambda_i|}, \quad \text{and} \quad \theta_i' = \frac{\lambda_i'}{|\lambda_i'|},$$

$i \in \{1,2\}$ where $\lambda_i(\lambda_i')$ are the eigenvalues of $A_1(A_2)$ and $\mu_i(\mu_i')$ are the eigenvalues of $h_1(h_2)$.

One can prove that there exists V_3, c^∞-open in V_2, such that if $g \in V_3$ and $h \in Z_F^w(g)$ then $\alpha_1 = \alpha_2 = \alpha_1' = \alpha_2'$.

Now if $g \in V_3'$ and $h \in Z_F^w(g)$ then $h' = h \circ g^i$ is identified with

$$(\sigma_1, \sigma_2, \sigma_1', \sigma_2', \alpha) \in (\mathbb{Z}_2)^4 \times \mathbb{R},$$

and

$$\begin{aligned}(\varphi_g^u)^{-1} \circ \varphi_g^s(\sigma_1^n \cdot \theta_1^k \cdot |\lambda_1|^{n\alpha+k} w_1,\ \sigma_2^n \cdot \theta_2^k \cdot |\lambda_2|^{n\alpha+k} w_2)\\ = ({\sigma_1'}^n \cdot {\theta_1'}^k \cdot |\lambda_1'|^{n\alpha'+k} w_1',\ {\sigma_2'}^n \cdot {\theta_2'}^k \cdot |\lambda_2'|^{n\alpha'+k} w_2'),\end{aligned}$$

for all $z \in W^s(P) \cap W^u(Q)$, where

$$(w_1, w_2) = (\varphi_f^s)^{-1}(z), (w_1', w_2') = (\varphi_f^u)^{-1}(z),$$

$$\sigma_i = \frac{\mu_i}{|\mu_i|} \quad \text{and} \quad \sigma_i' = \frac{\mu'}{|\mu_i'|}, i \in \{1,2\}.$$

Considering $h \circ f^{sk_0}$ instead of h (for a convenient s) we can assume that $\alpha \in]0,1]$; as before if $h \in Z_F^w(g)$ is such that

$$(\sigma_1, \sigma_2, \sigma_1', \sigma_2', \alpha) \neq (\theta_1, \theta_2, \theta_1', \theta_2', 1)$$

then h is called a *root of order* α. Now we can argue exactly as in Section 2:

- $V(\sigma_1, \sigma_2, \sigma_1', \sigma_2', \alpha) = \{f \in V_3\colon \text{equation } (*) \text{ is not satisfied}\}$ is a c^∞-open and dense subset of V_3;
- $V(1) \cap V(\frac{1}{2}) = (\cap_{\sigma_i, \sigma_i'} V(\sigma_1, \sigma_2, \sigma_1', \sigma_2', 1)) \cap (\cap_{\sigma_i, \sigma_i'} V(\sigma_1, \sigma_2, \sigma_1', \sigma_2', \frac{1}{2}))$ is c^∞-open and dense in V_3;

- there is $q_0 \in \mathbb{N}$ and V_4, c^∞-open and dense in $V(1) \cap V(\frac{1}{2})$, such that if $g \in V$ and $h \in \mathbb{Z}_F^w(g)$ is a root of order α then $\alpha = \frac{p}{q} \in \mathbb{Q}$, $(p,q) = 1$, and $q_0 > q > 2$.

Therefore if $g \in V_4$ then g can only have a finite number of roots and this number is uniformly bounded in V_4. This implies that there exists U, c^∞-open and dense in V_4, such that $Z_F^w(g)$ is trivial for all $g \in U$, thus ending the proof. □

4. Proof of Theorem 2

Let us now consider the one-parameter family

$$f_\rho : \mathbb{R}^2 \to \mathbb{R}^2, f_\rho(x,y) = (\rho - x^2 - y, x)$$

and prove that if $\rho > -1$ then $Z^w(f_\rho)$ is trivial.

To get this let us first state some properties of this family. If $\rho < -1$ then f_ρ has no fixed points; for $\rho = -1$ there exists one fixed point with eigenvalue 1. For $\rho > -1$ there are two fixed points P_ρ and Q_ρ, both in the diagonal, P_ρ is hyperbolic and $Df_\rho(P_\rho)$ has two real and positive eigenvalues, Q_ρ is an elliptic point for $\rho \in]-1,3[$, and if $\rho > 3$ then Q_ρ is hyperbolic and $Df_\rho(Q_\rho)$ has two real and negative eigenvalues.

As $R \circ f_\rho = f_\rho^{-1} \circ R$, $R(x,y) = (y,x)$, it follows that $R(W^s(P_\rho)) = W^u(P_\rho)$; also as the region U_0, respectively U_1, is f_ρ-invariant, respectively f_ρ^{-1}-invariant, (see the figure below), just two separatrixes can produce homoclinic points. Simetry along the diagonal and λ-*lemma* imply that there exists a primary (transversal) homoclinic point in the diagonal, say Y_q; from this fact and the analicity of $W^s(P_\rho)$ and of $W^u(P_\rho)$ on compact parts it is not difficult to prove that

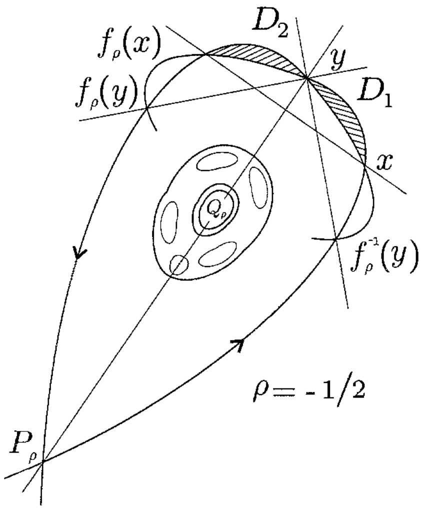
$$f_\rho^{-1}(\alpha_\rho) \cap \beta_\rho = \{Y_\rho, f_\rho^{-1}(Y_\rho), X_\rho\},$$

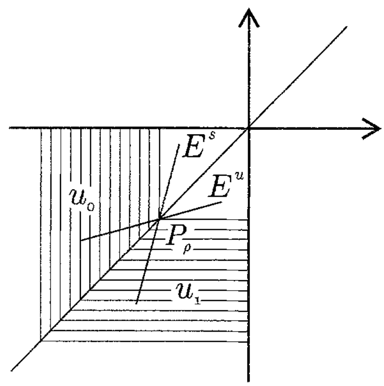
where α_ρ is the stable arc joining P_ρ and Y_ρ, and β_ρ is the unstable arc joining P_ρ and Y_ρ. Now if $h \in Z^w(f_\rho)$ then

$$h(P_\rho) = P_\rho, h(W^\sigma(P_\rho)) = W^\sigma(P_\rho), \sigma = s \text{ or } u,$$

and $Dh(P_\rho)$ has two real and positive eigenvalues (which implies that h preserves orientation.

Assume that h is not an integer power of f_ρ; from section two we have that $\alpha_h^s = \alpha_h^u \in \mathbb{Q} - \mathbb{Z}$. Moreover, as $\#\{f_\rho^{-1}(\alpha_\rho) \cap \beta_\rho\} = 3$, one has that $\alpha_h^s = \frac{s}{2}$, s odd. Considering $h \circ f_\rho^l$ instead of h, for a convenient l, we can assume that $\alpha_h^s = \frac{1}{2}$. This means that if $Z^w(f_\rho)$ is not trivial then there exists $h \in Z^w(f_\rho)$ such that $h^2 = f_\rho$, $h(X_\rho) = Y_\rho$ and $h(Y_\rho) = f_\rho(X_\rho)$. Therefore we have that $h(D_1) \subseteq D_2$, which is impossible since h must preserve orientation, has we have seen before. Thus $Z^w(f_\rho)$ must be trivial. □





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